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2001 J. Phys. A: Math. Gen. 34 L409

(http://iopscience.iop.org/0305-4470/34/30/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) L409–L415 PII: S0305-4470(01)22226-4

LETTER TO THE EDITOR

On the asymptotic behaviour of correlators of multi-cut matrix models

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Received 22 February 2001, in final form 14 June 2001
Published 20 July 2001
Online at stacks.iop.org/JPhysA/34/L409

Abstract

We consider invariant ensembles of $n \times n$ Hermitian random matrices (known also as the matrix models) in the case where the support of the limiting density of states (DOS) consists of several intervals. By using recent results on the asymptotics of orthogonal polynomials, we find first that the amplitudes of the leading terms of the correlator of the normalized traces of resolvent of random matrices and of their densities of states are quasi-periodic functions of n, whose frequencies are the integrals of the limiting DOS over the intervals of the support, and whose form is uniquely determined by the edges of the support. This suggest a certain parametrization of the universality classes of the correlator. Second we show that the leading terms of these correlators can be expressed correspondingly via the matrix elements of the resolvent and of the spectral kernel of a certain quasi-periodic Jacobi matrix whose coefficients are determined by the same frequencies.

PACS numbers: 02.30.Gp

In recent years the eigenvalue distribution of various ensembles of random matrices has been extensively studied being motivated by a number of questions in physics and mathematics (see recent works [5, 12–15, 19] and references therein). In particular, of considerable interest are unitary invariant ensembles of Hermitian matrices, known also as matrix models. Their probability distribution is defined by the density

$$p_n(M) = Z_n^{-1} \exp(-n \operatorname{Tr} V(M)) \tag{1}$$

with respect to the 'uniform' measure $\mathrm{d}M = \prod_{j=1}^{n} \mathrm{d}M_{jj} \prod_{j\leqslant k} \mathrm{d}\mathbb{R}\, M_{jk}\, \mathrm{d}\mathbb{I}\, M_{jk}$ in the space of Hermitian matrices $M = \{M_{jk}\}_{j,k}^{n}$, $M_{j,k} = \overline{M_{k,j}}$. In (1), Z_n^{-1} is the normalization constant and $V(\lambda)$ is a real-valued function, bounded from below and growing faster than $2\log|\lambda|$ as $|\lambda| \to \infty$. Typically V is a polynomial of degree $2p$ positive at infinity, although much broader classes of potentials can also be studied (see, for example, [6, 11, 21]).

One of the simplest basic characteristics of random matrices is their density of states (DOS):

$$\rho_n(\lambda) = n^{-1}\sum_{l=1}^{n}\delta(\lambda - \lambda_l^{(n)}) \tag{2}$$

where $\{\lambda_l^{(n)}\}_{l=1}^n$ are eigenvalues of a random matrix. Of particular interest are the DOS moments $\langle\rho_n(\lambda_1)\ldots\rho_n(\lambda_s)\rangle$ for any s-tuple $\{\lambda_1,\ldots,\lambda_s\}$, $s = 1,\ldots,n$. The symbol $\langle\cdots\rangle$ denotes here and below the averaging with respect to the probability distribution (1). In particular, if all λ_i are distinct, then [18]

$$\langle\rho_n(\lambda_1)\ldots\rho_n(\lambda_s)\rangle = n^{-s}R_{n,s}(\lambda_1,\ldots,\lambda_s) \tag{3}$$

where

$$R_{n,s}(\lambda_1,\ldots,\lambda_s) = \det\{K_n(\lambda_j,\lambda_k)\}_{j,k=1}^s \qquad K_n(\lambda,\mu) = \sum_{l=0}^{n-1}\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \tag{4}$$

$\{\psi_l^{(n)}(\lambda)\}_{l=0}^\infty$ is the orthonormalized system in which

$$\psi_l^{(n)}(\lambda) = \exp(-nV(\lambda)/2)P_l^{(n)}(\lambda) \tag{5}$$

and $\{P_l^{(n)}\}_{l=0}^\infty$ is the system of polynomials orthonormal with respect to the weight $w_n(\lambda) = \exp(-nV(\lambda))$. $K_n(\lambda,\mu)$ is called the reproducing kernel of the system $\{\psi_l^{(n)}(\lambda)\}_{l=0}^\infty$.

The simplest important cases of (3) are the mean density $\overline{\rho}_n$, and the (connected) DOS–DOS correlator $\kappa_n(\lambda_1,\lambda_2)$:

$$\overline{\rho}_n(\lambda) = \langle\rho_n(\lambda)\rangle \qquad \kappa_n(\lambda_1,\lambda_2) = \langle\rho_n(\lambda_1)\rho_n(\lambda_2)\rangle - \langle\rho_n(\lambda_1)\rangle\langle\rho_n(\lambda_2)\rangle. \tag{6}$$

It can be shown [6] (see also [22]) that under rather general conditions on V in (1) the DOS (2) tends weakly in probability to the non-random limit

$$\lim_{n\to\infty}\rho_n(\lambda) = \rho(\lambda) \tag{7}$$

that can be found as the unique minimizer of the electrostatic energy

$$\mathcal{E}[\rho] = \int V(\lambda)\rho(\lambda)\,\mathrm{d}\lambda - \iint \ln|\lambda-\mu|\rho(\lambda)\rho(\mu)\,\mathrm{d}\lambda\,\mathrm{d}\mu \qquad \int\rho(\lambda)\,\mathrm{d}\lambda = 1 \tag{8}$$

of linear charges subjected to the external field V.

In many cases the study of moments of the DOS is equivalent to that of the moments $\langle g_n(z_1)\ldots g_n(z_s)\rangle$ of the normalized traces of the resolvents of random matrices, i.e. the Stieltjes transforms of the DOS:

$$g_n(z) = n^{-1}\,\mathrm{Tr}(M-z)^{-1} = \int\frac{\rho_n(\lambda)}{\lambda - z}\,\mathrm{d}\lambda \qquad \mathbb{I}(z) \neq 0. \tag{9}$$

Most of the asymptotic results of the field concerns the case where the support σ of the limiting DOS ρ of (7) is a single interval. There are, however, interesting effects pertinent to the case where σ is a union of several disjoint intervals (it can be shown that if $\deg V = 2p$, then σ consists of p disjoint intervals at most). Thus in general we can write that

$$\sigma = \bigcup_{l=1}^{q}[a_l,b_l] \qquad -\infty < a_1 < b_1 < \cdots < a_q < b_q < \infty. \tag{10}$$

The multi-interval (multi-cut) case was studied in recent papers [1,3,7] (see also references on earlier works given in these papers). In particular, the large-n behaviour of the correlator

$\kappa_n(\lambda_1, \lambda_2)$ of (6) and of its double Stieltjes transform

$$\mathcal{D}_n(z_1, z_2) = \langle g_n(z_1) g_n(z_2)\rangle - \langle g_n(z_1)\rangle\langle g_n(z_2)\rangle \tag{11}$$

was discussed.

In this Letter, we would like to contribute to this discussion by using recent results [11,16] on the asymptotic form of orthogonal polynomials entering (5). Namely we show that, if the support of the limiting DOS consists of $q \geqslant 2$ disjoint intervals, then generically the amplitudes of the leading terms of $\mathcal{D}_n(z_1, z_2)$ and of $\kappa_n(\lambda_1, \lambda_2)$ are quasi-periodic functions of n, determined uniquely by $q-1$ incommensurate frequencies $(\alpha_1, \ldots, \alpha_{q-1})$ of (16) and by the edges of the support (10). This provides a certain parametrization of the leading terms, i.e. a parametrization of the classes of the long-range universality of matrix models. We also show that the correlator (11) can be expressed via matrix elements of the resolvent of a certain Jacobi matrix with quasi-periodic coefficients determined by the same frequencies.

We give first a convenient form of the correlator (11). This requires certain facts on orthogonal polynomials and associated Jacobi matrices (see, for example, [2]). System (5) gives rise to the semi-infinite Jacobi matrix

$$\mathcal{J}^{(n)} = \{J^{(n)}_{lm}\}^{\infty}_{l,m=1} \qquad J^{(n)}_{lm} = r^{(n)}_l \delta_{l+1,m} + s^{(n)}_l \delta_{l,m} + r^{(n)}_{l-1}\delta_{l-1,m} \qquad r^{(n)}_{-1} = 0 \tag{12}$$

via the recurrent relation $\lambda P^{(n)}_l(\lambda) = r^{(n)}_l P^{(n)}_{l+1}(\lambda) + s^{(n)}_l P^{(n)}_l(\lambda) + r^{(n)}_{l-1} P^{(n)}_{l-1}(\lambda)$, valid for $l \geqslant 0$ with $r^{(n)}_{-1} = 0$. The matrix $\mathcal{J}^{(n)}$ can be regarded as a self-adjoint operator acting in the space $l^2(\mathbb{Z}_+)$ of semi-infinite square integrable sequences. The resolution of identity $\mathcal{E}^{(n)}(\mathrm{d}\lambda) = \delta(\lambda - \mathcal{J}^{(n)})\,\mathrm{d}\lambda$ of this operator has the matrix elements $\mathcal{E}^{(n)}_{lm}(\mathrm{d}\lambda) = \psi^{(n)}_l(\lambda)\psi^{(n)}_m(\lambda)\,\mathrm{d}\lambda$, where $\psi^{(n)}_l(\lambda)$, $l = 0, 1, \ldots$ are defined in (5). In particular

$$R^{(n)}_{lm}(z) := (\mathcal{J}^{(n)} - z)^{-1}_{lm} = \int \frac{\psi^{(n)}_l(\lambda)\psi^{(n)}_m(\lambda)}{\lambda - z}\,\mathrm{d}\lambda. \tag{13}$$

Now, by using (2), (4), and (9), we can write the correlator $\mathcal{D}_n(z_1, z_2)$ as

$$\mathcal{D}_n(z_1, z_2) = \frac{1}{2n^2}\int\!\!\int \frac{(\lambda_1 - \lambda_2)^2 K_n(\lambda_1, \lambda_2)^2}{(\lambda_1 - z_1)(\lambda_1 - z_2)(\lambda_2 - z_1)(\lambda_2 - z_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2$$

or, in view of (13) and the Christoffel–Darboux identity [2], as

$$\mathcal{D}_n(z_1, z_2) = \frac{(r^{(n)}_{n-1})^2}{n^2(\delta z)^2}\left[\delta R^{(n)}_{n-1,n-1}.\delta R^{(n)}_{n,n} - (\delta R^{(n)}_{n,n-1})^2\right] \tag{14}$$

where for any function $f(z)$ we set $\delta f = f(z_1) - f(z_2)$. It is easy to verify that (14) is equivalent to the well known expression $\kappa_n\,(\lambda_1, \lambda_2) = -n^{-2}K^2_n(\lambda_1, \lambda_2)$ [18] for the correlator (6).

We recall now the results of paper [11] that we are going to use. Introduce the limiting integrated density of states (IDS) of random matrices

$$N(\lambda) = \int_{\lambda}^{\infty} \rho(\lambda')\,\mathrm{d}\lambda' \tag{15}$$

and the $(q-1)$-dimensional vector for $q \geqslant 2$:

$$\alpha = \{\alpha_l\}^{q-1}_{l=1} \qquad \alpha_l = N(a_{l+1}). \tag{16}$$

The numbers $\beta_l = \alpha_l - \alpha_{l+1},\ l = 1, \ldots, q-1$ can be interpreted as relative charges of the intervals $[a_l, b_l]$, $l = 2, \ldots, q$ of the support and are often called the filling numbers of these intervals.

According to [11], for any $n \in \mathbb{Z}$ and $\gamma = 0, 1$, there exist continuous in λ functions $\mathcal{N}_{n,\gamma}(\lambda)$, $\Gamma_{n,\gamma}(\lambda)$, and a number $0 < \tau \leqslant 1$, such that if λ belongs to the interior of the support σ (10), then

$$\psi^{(n)}_{n-\gamma}(\lambda) = \mathcal{N}_{n,\gamma}(\lambda)\cos(\pi n N(\lambda) + \Gamma_{n,\gamma}(\lambda)) + \mathcal{O}(n^{-\tau}) \qquad n \to \infty. \tag{17}$$

Moreover, $\mathcal{N}_{n,\gamma}(\lambda)$ and $\Gamma_{n,\gamma}(\lambda)$ depend on n via the vector $n\alpha$, i.e. there exist continuous in λ and in $x_1, \ldots, x_{q-1}$ functions $\mathcal{N}_\gamma(\lambda; x_1, \ldots, x_{q-1})$, and $\Gamma_\gamma(\lambda; x_1, \ldots, x_{q-1})$, periodic with period 1 with respect to each x_l, $l = 1, \ldots, q-1$ and such that

$$\mathcal{N}_{n,\gamma}(\lambda) = \mathcal{N}_\gamma(\lambda; x_1, \ldots, x_{q-1}) \qquad \Gamma_{n,\gamma}(\lambda) = \Gamma_\gamma(\lambda; x_1, \ldots, x_{q-1}). \tag{18}$$

The coefficient $r_{n-1}^{(n)}$ of the Jacobi matrix $\mathcal{J}^{(n)}$ of (12) has a similar asymptotic behaviour:

$$r_{n-1}^{(n)} = A(\alpha_1 n, \ldots, \alpha_{q-1} n) + \mathcal{O}(n^{-\tau}) \tag{19}$$

where $A(x_1, \ldots, x_{q-1})$ is a continuous function of period 1 with respect to each variable $x_1, \ldots, x_{q-1}$.

The functions $\mathcal{N}_\gamma,\ \Gamma_\gamma,\ \gamma = 0, 1$, and A can be expressed via the Riemann theta-function, associated in the standard way with two-sheeted Riemann surface obtained by gluing two copies of the complex plane slit along the gaps $(b_1, a_2), \ldots, (b_{q-1}, a_q), (b_q, a_1)$ of the limiting DOS support (10), the last gap goes through infinity. It is important that the functions $\mathcal{N}_\gamma,\ \Gamma_\gamma,\ \gamma = 0, 1$, and A are uniquely determined by the edges $a_1,\ b_1,\ a_2, \ldots, b_q$ of the support of the limiting DOS of the ensemble.

If λ belongs to the exterior of σ then each $\psi_{n-\gamma}^{(n)}$ decays exponentially in n as $n \to \infty$.

By using the facts described above, we can find the asymptotic form of the matrix elements (13) for $|\mathbb{I}z| \geqslant \eta > 0$, where η is independent of n:

$$R_{n-\gamma,n-\gamma}^{(n)}(z) = \frac{1}{2}\int_\sigma \frac{\mathcal{N}_{n,\gamma}^2(\lambda)}{\lambda - z}\,\mathrm{d}\lambda + \mathrm{o}(1) \qquad \gamma = 0, 1 \quad n \to \infty \tag{20}$$

$$R_{n-1,n}^{(n)}(z) = \frac{1}{2}\int_\sigma \frac{\mathcal{N}_{n,1}(\lambda)\mathcal{N}_{n,0}(\lambda)\cos(\Gamma_{n,1}(\lambda) - \Gamma_{n,0}(\lambda))}{\lambda - z}\,\mathrm{d}\lambda + \mathrm{o}(1) \qquad n \to \infty. \tag{21}$$

The formulae (14)–(21) lead to the following conclusions on the form of the amplitude $d_n(z_1, z_2)$ of the leading term of the correlator (11):

$$\mathcal{D}_n(z_1, z_2) = n^{-2} d_n(z_1, z_2)(1 + \mathrm{o}(1)) \qquad n \to \infty. \tag{22}$$

(i) In the generic case of incommensurable frequencies $\alpha_1, \ldots, \alpha_{q-1}$ of (16), $d_n(z_1, z_2)$ is a quasi-periodic function of n if $q \geqslant 2$.

(ii) $d_n(z_1, z_2)$ is uniquely determined by the edges $a_1,\ b_1,\ a_2, \ldots, b_q$ of the DOS support (10), and by the frequencies $\alpha_1, \ldots, \alpha_{q-1}$ of (16). Thus all the potentials V in (1) having the same set of these parameters leads to the same amplitude $d_n(z_1, z_2)$. By using the widely accepted terminology of the random matrix theory, we can say that the classes of universality with respect to the leading term of the correlator (known also as the classes of the long-range universality) are parametrized by $a_1,\ b_1,\ a_2, \ldots, b_q$ and by $\alpha_1, \ldots, \alpha_{q-1}$.

Similar conclusions are valid for the DOS–DOS correlator κ_n in (6). However, unlike the resolvent correlator (11) which becomes quasi-periodic as $n \to \infty$ for any fixed non-real z_1 and z_2, the DOS–DOS correlator gets a quasi-periodic universal form only after the integration with a smooth function $\phi(\lambda_1, \lambda_2)$ such that $\lim_{\lambda_1 \to \lambda_2}(\lambda_1 - \lambda_2)^{-2}\phi(\lambda_1, \lambda_2)$ is bounded (the weak or smoothed asymptotics).

This has to be contrasted with the short-ranged (or microscopic) universality that manifests itself in $1/n$-neighbourhoods of interior points of σ and is valid independently of the number of connected components of σ and under the rather general conditions on the potential V in (1) (see papers [11, 21]). Thus under conditions of these papers all the unitary invariant ensembles (1) belong to the same short-range universality class. On the other hand, since according to the above the form of the leading terms of the covariance $\mathcal{D}_n(z_1, z_2)$ depends on the number of the intervals of the DOS support, the long-range universality is more sensitive to the form of the potential (see, for example, formulae (26) and (27) below, corresponding to the one-interval case and the two-interval case of a symmetric potential).

We remark also that similar periodic and quasi-periodic modulations of certain asymptotic formulae have already appeared in mathematical physics. We mention here the asymptotic form of the DOS of the random discrete operators near their spectrum edges [17], and of the Fredholm determinants of the integral operator determined by the kernel $\sin \pi(x-y)/\pi(x-y)$ on the union of disjoint intervals of the real axis when the total length of the intervals tends to infinity [10].

We will argue now that the leading terms of the matrix elements (20), (21) of $R^{(n)}(z) = (\mathcal{J}^{(n)} - z)^{-1}$ coincide asymptotically with the matrix elements of the resolvent of a certain quasi-periodic matrix. For the sake of simplicity we restrict ourselves to the case of an even function V in (1), when the coefficients $s_l^{(n)}$ in (12) is zero. We need one more result concerning the asymptotic behaviour of the entries $r_l^{(n)}$ of the matrix $\mathcal{J}^{(n)}$ of (12). Replace V in (1) by V/g, $0 < g < g_0 < \infty$. Then, according to [11, 16], for all g, except for a possible finite set of values, there exist a continuous function $A(x_1, \dots, x_{q-1}; g)$ of period 1 with respect to each variable $x_1, \dots, x_{q-1}$, and $\epsilon > 0$ such that uniformly in $g' \in [g-\epsilon, g+\epsilon]$

$$r_{n-1}^{(n)}(g') = A(n\alpha; g') + \mathcal{O}(n^{-\tau}) \qquad n \to \infty. \tag{23}$$

Since $A(x; g)$ is periodic in x, it can be considered as defined on the torus $\mathbb{T}^{q-1}$. Suppose first that the numbers $\alpha_1, \dots, \alpha_{q-1}$ are rationally independent (incommensurate). Then for any $x \in \mathbb{T}^{q-1}$, there exists a subsequence $\{n_j\}$, such that the sequence of vectors $\{\{n_j\alpha\} \in \mathbb{T}^{q-1}\}$, whose components are the fractional parts of the components of the vectors $n_j\alpha$, converges to x as $n_j \to \infty$. Consider now the sequence $a_j = n_j + k,\ k = \mathrm{o}(n_j),\ n_j \to \infty$. Then, according to (1) and (23), we have $r_{a_j-1}^{(n_j)}(g) = r_{a_j-1}^{(a_j)}(a_j/n_j\, g) = A(n_j\alpha + k\alpha; (1+k/n_j)g) + \mathcal{O}(1/n^\tau) = r_k^{[x]}(g) + \mathrm{o}(1),\ n_j \to \infty$, where

$$r_k^{[x]}(g) = A(x + k\alpha; g). \tag{24}$$

This allows us to introduce the double infinite Jacobi matrix $\mathcal{J}^{[x]}$ whose off-diagonal entries are $r_k^{[x]}(g),\ k \in \mathbb{Z}$, the diagonal entries are zero and the spectrum is σ. For any $x \in \mathbb{T}^{q-1}$, the entries $r_k^{[x]}$ are quasi-periodic in k.

By using (23), (24) and the resolvent identity for the pair $(\mathcal{J}^{(n)}, \mathcal{J}^{[x]})$, it can be shown that the matrix elements $R_{a_jb_j}^{(n)}(z) = ((\mathcal{J}^{(n)} - z)^{-1})_{a_jb_j}$ with $a_j = n_j + k,\ b_j = n_j + l$ coincide at the limit $n_j \to \infty,\ k, l = \mathrm{o}(n_j)$ with the matrix elements $R_{kl}^{[x]}(z) = ((\mathcal{J}^{[x]} - z)^{-1})_{kl}$. This leads to the following expression for the amplitude (22) of the leading term of the correlator (11):

$$d^{[x]}(z_1, z_2) = \frac{(r_0^{[x]})}{(\delta z)^2}\left(\delta R_{00}^{[x]} \delta R_{11}^{[x]} - \left(\delta R_{01}^{[x]}\right)^2\right). \tag{25}$$

The formula demonstrates once more the quasi-periodic dependence of the leading term of the correlator (11) on n via its dependence on x and allows us to find, at least in principle, the leading term of $\mathcal{D}_n$ up to its dependence of the 'initial phase' by computing the matrix elements of the resolvent $R^{[x]}$.

Note that the matrix $\mathcal{J}^{[x]}$ is similar to the finite-band quasi-periodic Jacobi matrices appearing in the integration of the nonlinear Toda lattice equations [23]. The similarity is in the procedure of the construction of the Riemann surface and respective Riemann theta-functions from a given spectrum and the difference is that the role of the (generalized) quasi-momentum of the Toda theory here plays the IDS (15) of random matrices multiplied by π (recall, that the quasi-momentum divided by π is the IDS of quasi-periodic Jacobi matrices, see, for example, [20]). The difference disappears for the class of polynomial potentials described at the end of this Letter (see also [9]).

We have discussed above the case when all frequencies $\alpha_l,\ l = 1, \dots, q-1$ are incommensurate. This case may be viewed as generic (see, for example, the recent paper [16]).

In a non-generic case, where the components of α are rationally dependent, the set of values of the parameter x, indexing the limiting Jacobi matrix $\mathcal{J}^{[x]}$, can be obtained as the closure of all limiting point of the sequence $\{n\alpha\}_{n\geqslant 0}$. In the theory of almost-periodic functions, this set is known as the hull of the almost periodic functions with a given set of frequencies in its generalized Fourier series. An interesting non-generic case is where all the frequencies are integer multiples of $1/p$, where $p \geqslant 2$ is a positive integer. In this case the index set for matrices $J^{[x]}$ is the finite set $\{x = m/p; m = 0, 1, \ldots, p-1\}$ instead of the whole torus $\mathbb{T}^{q-1}$ in the generic case, and the amplitude d_n of the leading term of (25) is p-periodic in x.

Consider the two simplest cases: (1) of the one-interval support and; (2) of the of two-interval support of an even potential: $V(\lambda) = V(-\lambda)$. In the first case the entries of $\mathcal{J}^{[x]}$ are independent of the matrix index and of x and assuming without loss of generality that $\sigma = [-a, a]$ for some $a > 0$, we obtain that $r_k^{[x]} = a/2$, $s_k^{[x]} = 0$. We obtain the x-independent leading term amplitude

$$d(z_1, z_2) = -\frac{1}{2(z_1 - z_2)^2}\left[1 - \frac{(z_1 z_2 - a^2)}{X_1^{1/2}(z_1, z_2)}\right] \tag{26}$$

where $X_1(z_1, z_2) = (z_1^2 - a^2)(z_2^2 - a^2)$. In the second case we have $\sigma = [-b, -a] \cup [a, b]$ for some $0 < a < b < \infty$ and the matrix $\mathcal{J}^{[x]}$ is 2-periodic. The diagonal entries of $\mathcal{J}^{[x]}$ are zero and the off-diagonal entries are either $(b - (-1)^{k+m}a)/2$ or $(b + (-1)^{k+m}a)/2$, $m = 0, 1$, i.e. defined by the spectrum up to the initial phase (similar ambiguity was found in [7]). By using the results of [11], it can be shown that the first possibility is the case and we obtain for the 2-periodic in m leading term

$$d^{[x]}(z_1, z_2) = -\frac{1}{2(z_1 - z_2)^2}\left[1 - \frac{(z_1 z_2 - a^2)(z_1 z_2 - b^2)}{X_2^{1/2}(z_1, z_2)}\right] - \frac{(-1)^m ab}{2X_2^{1/2}(z_1, z_2)} \tag{27}$$

where $X_2(z_1, z_2) = (z_1^2 - a^2)(z_1^2 - b^2)(z_2^2 - a^2)(z_2^2 - b^2)$.

Expression (26) agrees with that obtained in [4, 8] by other methods, and expression (27) agrees with that obtained in [7] and differs from that found in [1, 3], where the analogue of this expression is independent of x and contains elliptic integrals whose arguments are determined by a and b, while expression (27) is an elementary function of all its arguments but is 2-periodic in m. It can be shown that in a general case of a two-interval but not necessarily symmetric potential the leading term is a quasi-periodic in m and contains the Jacobi elliptic functions (but not the elliptic integrals as in [1, 3]). The elliptic functions disappear when one passes to a two-interval symmetric potential.

In conclusion we will give a class of potentials always leading to the periodic Jacobi matrices [9], thus for the periodic in x leading coefficient. For any positive integer p take a polynomial $v(\lambda)$ of degree p with real coefficients such that $v(\lambda) = \lambda^p + \mathcal{O}(\lambda^{p-1})$, $\lambda \to \infty$ and that for some $g > 0$ all the zeros of the polynomial $v^2(\lambda) - 4g$ are real and simple. If there exist a constant C such that the potential can be written in the form $V(\lambda) = v^2(\lambda)/(2p) + C$, then the limiting DOS is $\rho(\lambda) = (2\pi pg)^{-1/2}|v'(\lambda)||4g - v^2(\lambda)|^{1/2}\mathbf{1}_\sigma(\lambda)$, its support consists of p intervals (determined by the zeros of $v^2(\lambda) - 4g$) and $\alpha_l = l/p$, $l = 0, \ldots, p-1$, i.e. the Jacobi matrix $J^{[x]}$ is p-periodic. The proof of a given statement will be published elsewhere [9].

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